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Journal of Approximation Theory 139 (2006) 371–401

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

The Euler–Lagrange Theory for Schur's algorithm: Wall pairs

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Received 25 February 2005; accepted 10 October 2005

Communicated by Guillermo López Lagomasino
Available online 28 November 2005

To Barry Simon on the occasion of the 60th Birthday

Abstract

This paper develops a techniques of Wall pairs for the study of periodic exposed quadratic irrationalities in the unit ball of the Hardy algebra.

MSC: 11A55; 30B60; 30B70; 30D50; 30D55; 42C05

Keywords: Schur's algorithm; Periodic Schur's functions; Wall continued fractions; Wall pairs

1. Introduction

1. Regular continued fractions

$$\xi = b_0 + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{1}{b_k}\right) = b_0 + \frac{1}{b_1} \, {}_{+\cdots+} \, \frac{1}{b_n + \xi_n} \tag{1}$$

parameterize the continuum $\mathbb{R}$ of all real numbers. Another important continuum, the unit ball $\mathcal{B}$ of the Hardy algebra H^∞, can be parameterized by Schur's parameters $\{a_n\}_{n\geqslant 0}$ via Wall continued fractions [12]

$$f(z) = a_0 + \frac{(1-|a_0|^2)z}{\overline{a_0}z} \, {}_{+} \, \frac{1}{a_1} \, {}_{+} \, \frac{(1-|a_1|^2)z}{\overline{a_1}z} + \cdots. \tag{2}$$

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2005.10.001

Comparing these two parameterizations, one can transfer theories related with regular continued fractions to those related to Wall continued fractions. In [5] using this approach, we established some properties of algebraic, and in particular quadratic, exposed irrationalities.

Continuum $\mathcal{B}$ consists of all holomorphic mappings of the unit disc $\mathbb{D}$ into itself. Assuming that $f = f_0 \in \mathcal{B}$ and applying Schwarz' lemma, we obtain a sequence $\{f_n\}_{n\geqslant 0}$ of functions in $\mathcal{B}$ and a sequence $\{a_n\}_{n\geqslant 0}$ of complex numbers in $\mathbb{D}$ such that

$$f_0 = \frac{zf_1(z) + a_0}{1 + \bar{a}_0 z f_1(z)}; \dots; \ f_n = \frac{zf_{n+1}(z) + a_n}{1 + \bar{a}_n z f_{n+1}(z)}; \dots . \tag{3}$$

Schur's algorithm (3) terminates at the nth step if $|a_n| = 1$ and runs up to infinity if $|a_n| < 1$, $n = 0, 1, \dots$. It is clear that (3) terminates if and only if f is a rational function in $\mathcal{B}$ satisfying $|f| = 1$ on $\mathbb{T}$. Such functions are called *finite Blaschke products* [2]. The complex numbers a_n are called the *Schur parameters* of f. The functions f_n are called the *Schur functions* (of order n) associated with f. Elementary algebra shows that

$$f_n(z) = \frac{zf_{n+1}(z) + a_n}{1 + \overline{a_n} z f_{n+1}(z)} = a_n + \frac{(1 - |a_n|^2)z}{\overline{a_n} z + 1/f_{n+1}(z)}.$$

Iterations lead to the Wall continued fraction (2). An Wall continued fraction is finite if and only if it represents a finite Blaschke product. Comparing (1) and (2), we conclude that finite Blaschke products in $\mathcal{B}$, which are finite continued fractions (2) play the same role as finite continued fractions (1), i.e. $\mathbb{Q}$ in $\mathbb{R}$. Caratheodory's classical theorem states that the set of finite Blaschke products is dense in $\mathcal{B}$.

Theorem 1.1 (*Carathéodory*). *If $f \in \mathcal{B}$, then there is a sequence $\{\beta_n\}_{n\geqslant 0}$ of finite Blaschke products which converges to f in the $*$-weak topology of $\mathcal{B}$.*

The classical proof (see [2, Ch. I, §2, Theorem 2.1]) gives more than it is stated in Theorem 1.1. Namely, the sequence $\{\beta_n\}_{n\geqslant 0}$ can be chosen to satisfy

$$f(z) = \beta_n(z) + O(z^n), \quad z \to 0.$$

2. However, opposite to the number field case, where any rational number is a convergent to a regular continued fraction, the convergents of Wall continued fractions (2) have a very special structure. The convergent A_n/B_n of order $2n$ for (2) is obtained by terminating (2) at the partial fraction $1/a_n$ followed by a finite number of arithmetic operations indicated in (2) without cancellations. It can be easily checked by induction that A_n, B_n are polynomials in $\mathbb{C}_n[z]$ and therefore the polynomials

$$A_n^*(z) = z^n \overline{A\left(\frac{1}{\bar{z}}\right)}, \quad B_n^*(z) = z^n \overline{A\left(\frac{1}{\bar{z}}\right)}$$

are also in $\mathbb{C}_n[z]$. Then the convergent of order $2n+1$ for (2) is given by zB_n^*/zA_n^*, see e.g. [4, Lemma 4.1].

The polynomials A_n, B_n are called the *Wall polynomials* associated with a sequence of parameters $\{a_n\}_{n\geqslant 0}$. Using matrix representations for numerators and denominators of convergents for general continued fractions [8, Ch. I, §5, (11–12)], one can obtain the following identity:

$$\begin{pmatrix} zB_n^* & -A_n^* \\ -zA_n & B_n \end{pmatrix} = \prod_{k=0}^{n} \begin{pmatrix} z & -\bar{a}_n \\ -a_n z & 1 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} z & -\bar{a}_n \\ -za_n & 1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} z & -\bar{a}_0 \\ -a_0 z & 1 \end{pmatrix}, \tag{4}$$

which can also be presented in the form of a system of linear recurrences

$$\begin{aligned} B_{n+1}^* &= zB_n^* + \bar{a}_{n+1}A_n, & A_{n+1}^* &= zA_n^* + \bar{a}_{n+1}B_n, \\ A_{n+1} &= A_n + a_{n+1}zB_n^*, & B_{n+1} &= B_n + a_{n+1}zA_n^*. \end{aligned} \tag{5}$$

These explicit formulas are especially useful when $a_n = 0$ for some n's, see [4, §§3–4] for details. The following *determinant* identity:

$$B_n^* B_n - A_n^* A_n = z^n \prod_{k=0}^{n} (1 - |a_k|^2) \stackrel{def}{=} \omega_n z^n \tag{6}$$

is an easy corollary of (4). It is obtained by application of the multiplicative functional $C \to \det(C)$ to both sides of (4). The equality $B_n(0) = 1$ and (6) imply that the polynomials A_n and B_n are relatively prime. Being restricted to $\mathbb{T}$, (6) takes an equivalent form $|B_n|^2 - |A_n|^2 \equiv \omega_n$, $z \in \mathbb{T}$.

2. The Wall pairs

3. Wall pairs are numerators and denominators of the even convergents to (2).

Definition 2.1. A pair (A, B) of relatively prime polynomials is called an Wall pair *if there are an Wall continued fraction* (2) *and an integer* $n \in \mathbb{N}$ *such that* $A_n = A$, $B_n = B$.

Theorem 2.1. *A pair* (A, B) *of relatively prime polynomials is an Wall pair if and only if the following conditions hold*:

(a) $\deg A \geqslant \deg B$;
(b) $B(0) = 1$ *and* $\inf\{|B(z)| : z \in \mathbb{D}\} > 0$;
(c) *there exists a positive constant* ω *such that*

$$|B|^2 - |A|^2 \equiv \omega, \quad z \in \mathbb{T}. \tag{7}$$

Proof. Suppose first that $A = A_n$, $B = B_n$ are defined by (4) and by a finite sequence of parameters $\{a_k\}_{k=1}^n$. Without loss of generality we may assume that $a_0 \neq 0$, since otherwise we can divide A_n by the corresponding power of z. Since $A_0 = a_0$, $B_0 = 1$, we obtain by induction (see details in [4, Corollary 4.3])

$$A_n = a_0 + \cdots + a_n z^n, \quad B_n = 1 + \cdots + \bar{a}_n a_0 z^n, \quad n = 1, 2, \ldots .$$

Observing that by (5) $A_n = A_{n-1}$, $B_n = B_{n-1}$ if $a_n = 0$, we conclude that $\deg A_n = \deg B_n = n$ if $a_n \neq 0$ and $\deg A_n = \deg B_n < n$ if $a_n = 0$. The fact that B_n satisfies (b) can be proved by induction (see [4, Lemma 4.5]). Finally, (c) is nothing but the restriction of the determinant identity (6) to $\mathbb{T}$.

To prove the converse statement we assume that it holds for polynomials of degree smaller than n. Let A and B be polynomials satisfying (a)–(c) and let $\deg A = n$. Then

$$A = az^n + \cdots + a_0, \quad a \neq 0; \quad B = bz^n + \cdots + 1. \tag{8}$$

Since the polynomials A and B satisfy (c), we can write on $\mathbb{T}$:

$$\omega z^n = z^n \bar{B} B - z^n \bar{A} A = B^* B - A^* A = (b - \bar{a}_0 a) z^{2n} + \cdots .$$

It follows that

$$b - \bar{a}_0 a = 0. \tag{9}$$

By the maximum modulus principle we obtain from (b) and (c) that $A/B \in \mathcal{B}$. Moreover, it follows from (c) that

$$\max_{\mathbb{T}} \left| \frac{A}{B} \right| < 1, \tag{10}$$

which implies that $|a_0| < 1$. Applying Schur's algorithm to $f = A/B$, we can consider the Schur function f_1:

$$f_1 = \frac{f - a_0}{z(1 - \bar{a}_0 f)} = \frac{(A - a_0 B)/z}{B - \bar{a}_0 A} = \frac{A'}{B'}, \tag{11}$$

where the polynomials A' and B' are defined by

$$A' = \frac{A - a_0 B}{z(1 - |a_0|^2)}, \quad B' = \frac{B - \bar{a}_0 A}{1 - |a_0|^2}.$$

It follows from (8) and (9) that

$$A' = a z^{n-1} + \cdots, \quad \deg B' \leqslant n - 1.$$

It is clear from the definition that $B'(0) = 1$ and from (10) we obtain that B' does not vanish in $\mathrm{Clos}(\mathbb{D})$. Next, we have on $\mathbb{T}$

$$|B'|^2 - |A'|^2 = \frac{|B - \bar{a}_0 A|^2 - |A - a_0 B|^2}{(1 - |a_0|^2)^2} = \frac{|B|^2 - |A|^2}{1 - |a_0|^2} = \frac{\omega}{1 - |a_0|^2},$$

which implies that $B^{'*} B' - A^{'*} A' = z^{n-1} \omega (1 - |a_0|^2)^{-1}$. It follows that A' and B' are relatively prime polynomials. Hence by the induction hypothesis we conclude that (A', B') is an Wall pair. By (11) and by definition of Schur's algorithm, we obtain that (A, B) is an Wall pair. $\Box$

Remark 2.1. Property (a) follows from (c) and (b). Indeed, since $B(0) = 1$ the trigonometric polynomial $|B|^2 = B\overline{B}$ has exactly degree $\deg(B)$. On the other hand by (c) $|B|^2 = \omega + A\overline{A}$, which proves the claim, since $A = z^k A_0$, where $A_0(0) \neq 0$. Thus, Wall pairs are completely characterized by properties (b) and (c).

Remark 2.2. Observe that for every Wall pair we have $\omega \in (0, 1)$. This follows directly from the formula

$$\omega = \prod_{k=0}^{n} (1 - |a_k|^2),$$

or can be obtained by the mean-value theorem applied to the harmonic function $\log |B|$:

$$\log \omega = \int_{\mathbb{T}} \log \omega \, dm < \int_{\mathbb{T}} \log |B|^2 \, dm = \log |B(0)|^2 = 0,$$

see (7).

Remark 2.3. The equality $\deg A = \deg B + s$, $s > 0$ takes place if and only if the first s parameters of A/B are zeros.

Remark 2.4. If (A, B) is an Wall pair, then $(-A^*, B)$ also is an Wall pair with the same ω.

4. The following theorem claims the leading role of A in (A, B).

Theorem 2.2. *In order that a polynomial A be the first component of an Wall pair (A, B) it is necessary and sufficient that*

$$\int_{\mathbb{T}} \log |A| \, dm < 0. \tag{12}$$

The second component B is uniquely determined by A.

Proof. If (A, B) is an Wall pair then $A/B \in \mathcal{B}$ and moreover (10) holds. Together with the mean-value theorem applied to the harmonic function $\log |B|^2$ this implies that

$$\int_{\mathbb{T}} \log |A|^2 \, dm = \int_{\mathbb{T}} \log \left| \frac{A}{B} \right|^2 dm + \log |B(0)|^2 < 0.$$

Suppose now that (12) holds and consider an auxiliary function

$$I(\omega) = \int_{\mathbb{T}} \log(|A|^2 + \omega) \, dm \tag{13}$$

on $[0, +\infty)$. It is clear that I is continuous and $I(0) < 0$, $I(1) > 0$. Since I is an increasing function, there exists a unique root ω in $(0, 1)$ of the equation $I(\omega) = 0$. By Fejér's theorem [11, Theorem 1.2.2] there exists a unique algebraic polynomial B such that $\deg B \leqslant \deg A$, $B(0) > 0$, B does not vanish in $\mathrm{Clos}(\mathbb{D})$ and

$$|B|^2 = |A|^2 + \omega$$

on $\mathbb{T}$. By the mean-value theorem we have

$$\log B(0)^2 = \int_{\mathbb{T}} \log(|A|^2 + w) \, dm = I(\omega) = 0$$

due to our choice of ω. Hence $B(0) = 1$ and (A, B) is an Wall pair by Theorem 2.1. □

It is interesting that Theorem 2.2 can be stated in purely geometric terms.

Corollary 2.3. *A polynomial A is the first component of an Wall pair (A, B) if and only if there exists $p > 0$ such that*

$$\int_{\mathbb{T}} |A|^p \, dm < 1. \tag{14}$$

Proof. Combining a well-known formula [2, Ch. IV, §6, Exers. 6-c]

$$\lim_{p \to 0+} \left(\int_{\mathbb{T}} |A|^p \, dm \right)^{1/p} = \exp \left\{ \int_{\mathbb{T}} \log |A| \, dm \right\} \tag{15}$$

with Theorem 2.2, we obtain the result. □

5. Let $\overset{\circ}{\mathcal{B}}_p$ be the open unit ball of the Hardy space H^p in $\mathbb{D}$ [2, Ch. II]. Then by Corollary 2.3 the set of the first components of Wall pairs in $\mathbb{C}_n[z]$ coincides with

$$\mathcal{A}_n \overset{def}{=} \mathbb{C}_n[z] \cap \bigcup_{p>0} \overset{\circ}{\mathcal{B}}_p .$$

It follows that $\mathcal{A}_n$ is an open subset of $\mathbb{C}_n[z]$ endowed with the Euclidean topology. If $A \in \mathcal{A}_n$, then the unique polynomial $B = \mathcal{W}(A)$, such that (A, B) is an Wall pair, can be given explicitly by

$$B = \mathcal{W}(A) = \exp\left\{\frac{1}{2}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(|A(\zeta)|^2 + \omega)\, dm(\zeta)\right\}, \tag{16}$$

where ω is the unique root of $I(\omega) = 0$ (see (13)). The mapping $\mathcal{W}$ is of C^∞-class on $\mathcal{A}_n$.

Theorem 2.4. *Let B be an arbitrary polynomial such that $B(0) = 1$ and*

$$0 < \omega \leqslant \inf\{\,|B(z)|^2 : |z| \leqslant 1\,\}. \tag{17}$$

Then for every ω satisfying (17) *there exists a polynomial A, not vanishing in $\mathbb{D}$, such that $|B|^2 - |A|^2 \equiv \omega$ on $\mathbb{T}$, $A(0) > 0$, and (A, B) is an Wall pair.*

Proof. By Fejér's theorem [11, Theorem 1.2.2] we can define the polynomial A by the following formula:

$$A(z) = \exp\left\{\frac{1}{2}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log\left(|B(\zeta)|^2 - \omega^2\right) dm(\zeta)\right\}, \quad |z| < 1, \tag{18}$$

which completes the proof by Theorem 2.1. □

Definition 2.2. Two Wall pairs (A, B) and $(\tilde{A}, \tilde{B})$ are called equivalent *if* $A^*A = \tilde{A}^*\tilde{A}$ *and* $B = \tilde{B}$.

By Theorem 2.1(c) $\omega = \tilde{\omega}$ for equivalent pairs. For equivalent pairs with given B and $\omega > 0$, the polynomial defined by (18) is called *maximal*. The reason for such a terminology is seen from the following theorem.

Theorem 2.5. *Let B, $B(0) = 1$, be a polynomial with ω, satisfying* (17). *Then the maximal polynomial is the unique polynomial, which maximizes the value $\Re A(0)$ taken over the first components of the equivalent pairs with fixed B and ω.*

Proof. Let $\overset{\circ}{A}$ be the maximal polynomial defined by (18) and let Λ be the set of roots of $\overset{\circ}{A}$ counting the multiplicities. It follows from (18) that $|\lambda| \geqslant 1$ for every $\lambda \in \Lambda$. Then

$$\overset{\circ}{A}{}^*\overset{\circ}{A} = |c|^2 \prod_{\lambda\in\Lambda} (z - \lambda)(1 - \bar{\lambda}z),$$

where $0 < \overset{\circ}{A}(0) = c\prod_{\lambda\in\Lambda}(-\lambda)$. It follows that all other polynomials A satisfying $A^*A = \overset{\circ}{A}{}^*\overset{\circ}{A}$ with $A(0) > 0$ are obtained from $\overset{\circ}{A}{}^*$ by multiplying it with a finite Blaschke product, which makes the value at the origin smaller. □

For any polynomial B, satisfying (b) of Theorem 2.1, and for any ω, $0 < \omega \leqslant \inf_{\mathbb{T}} |B|^2$ there exist not more than 2^n, $n = \deg B$, equivalent pairs (A, B) with given ω and $A(0) > 0$.

Corollary 2.6. *The set $\mathcal{A}_n$ is a bounded subset of $\mathbb{C}_n[z]$.*

Proof. Since every polynomial in $\mathcal{A}_n$ is a product of a finite Blaschke product and a maximal polynomial, it is sufficient to prove that

$$\int_{\mathbb{T}} |A|^2 \, dm = \sum_{k=0}^{n} |c_k|^2 \leqslant C_n \tag{19}$$

for every maximal polynomial $A = c(z - \lambda_1) \cdots (z - \lambda_n)$, C_n being a constant depending on n. By Theorem 2.2 we have $|c\lambda_1 \cdots \lambda_n| = |A(0)| < 1$. Combining this inequality with $|\lambda_j| \geqslant 1$, $j = 1, \ldots, n$, which are valid for any maximal polynomial, we obtain that

$$|c| \leqslant |c\lambda_1 \cdots \lambda_n| < 1. \tag{20}$$

The coefficient c_k is the products of the kth elementary symmetric function of $\lambda_1, \ldots, \lambda_n$ and c. The number of terms in this symmetric function is $\binom{n}{k}$ and each term is the product of k roots with c. By (20) this product cannot exceed 1. Thus,

$$\sum_{k=0}^{n} |c_k|^2 \leqslant \left(\sum_{k=0}^{n} |c_k| \right)^2 \leqslant \left(\sum_{k=0}^{n} \binom{n}{k} \right)^2 = 4^n. \qquad \square$$

Notice the essential difference between A and B in an Wall pair (A, B). The first component A is an interior point of $\mathcal{A}_n$, whereas the second B lies on the boundary $\partial \mathcal{A}_n$. Indeed, for B we have

$$0 = \log |B(0)|^2 = \int_{\mathbb{T}} \log |B|^2 \, dm,$$

and for A

$$\int_{\mathbb{T}} \log |A|^2 \, dm < 0.$$

Definition 2.3. The degree d of an Wall pair (A, B) is defined as $d = \deg A$. Every pair (A, B) is determined by the rational fraction $A/B \in \mathcal{B}$ with Schur's parameters $a_0, a_1, \ldots, a_d, 0, 0, \ldots$. The parameters $\{a_0, a_1, \ldots, a_d\}$ are called the parameters of the pair (A, B).

Notice that the parameters of $(-A^*, B)$ are given by $-\bar{a}_d, -\bar{a}_{d-1}, \ldots, -\bar{a}_0$.

3. Euler's theorem for Schur's algorithm

6. A regular continued fraction is called *periodic* if there exist $h \in \mathbb{Z}$, $h \geqslant 0$ and $d \in \mathbb{N}$ such that $b_{j+d} = b_j$ for $j = h, h+1, \ldots$. If $h = 0$, then a periodic continued fraction is called *purely periodic*. The first step here is due to Euler who observed that periodic regular continued fractions represent quadratic irrationalities.

The periodicity of quadratic irrationalities in $\mathbb{R}$ is stated in terms of associated continued fractions. In case of $\mathcal{B}$ these are Wall continued fractions (=Schur's algorithm). By definition periodic continued fractions cannot terminate. It follows that the parameters $\{a_n\}_{n\geqslant 0}$ must satisfy $|a_n| < 1, n = 0, 1, \dots$.

Definition 3.1. An Wall continued fraction (2) (equivalently Schur's algorithm (3)) is called periodic *if*

$$\begin{aligned} |a_n| &< 1, \quad n = 0, 1, \dots, \\ a_{j+L} &= a_j, \quad j = h, h+1, \dots \end{aligned}$$

for some integers $L, L > 0$ and $h, h \geqslant 0$. If $h = 0$, then we say that the continued fraction is purely periodic.

To stress the similarity with the Euler–Lagrange classical theory of quadratic irrationalities in $\mathbb{R}$ we use the notation

$$f = \{a_0, \dots, a_{h-1}, \overline{a_h, a_{h+1}, \dots a_{h+L-1}}\}$$

for Wall periodic continued fractions (Schur's Algorithm) corresponding to $f \in \mathcal{B}$. Every Wall pair determines a quadratic irrationality in $\mathcal{B}$. We consider a more general construction fixing an arbitrary element $\beta \in \mathcal{B}$.

Theorem 3.1. *Let (A, B) be an Wall pair of degree d, $d \geqslant 0$, and $\beta \in \mathcal{B}$. Then there exists a unique function f in $\mathcal{B}$ such that*

(a) *the parameters $a_0, a_1, \dots a_d$ of f and A/B coincide*, *i.e. A/B is the convergent to f of order $2d$*;
(b) *the $(d+1)$th Schur function f_{d+1} of f satisfies*

$$f_{d+1} = \beta f; \tag{21}$$

(c) *the function f satisfies the following quadratic equation*:

$$zA^*\beta X^2 + (B - zB^*\beta)X - A = 0. \tag{22}$$

Proof. Let $h^{(0)} = A/B$, and $\{h^{(n)}\}_{n\geqslant 0}$ be the sequence in $\mathcal{B}$ defined by

$$h^{(n+1)}(z) = \frac{A + zB^*\beta h^{(n)}(z)}{B + zA^*\beta h^{(n)}(z)}, \quad z \in \mathbb{D}. \tag{23}$$

Since $B^*B - A^*A = z^d\omega$ for (A, B), we obtain by (23) that

$$h^{(n+1)}(z) - h^{(n)}(z) = \frac{\omega z^{d+1}\beta(h^{(n)} - h^{(n-1)})}{(B + zA^*\beta h^{(n)})(B + zA^*\beta h^{n-1})}.$$

The iteration of the above formula yields

$$h^{(n+1)}(z) - h^{(n)}(z) = \frac{(\omega\beta)^{n+1} z^{n(d+1)} z^{d+1} h^{(0)}}{(B + zA^*\beta h^{(n)})(B + zA^*\beta h^{(n-1)})^2 \cdots (B + zA^*\beta h^{(0)})^2},$$

which obviously implies the convergence of $\{h^{(n)}\}_{n\geqslant 0}$ in the topology of the ring $\mathbb{C}[[z]]$ of formal Taylor series at $z = 0$. Since $h^{(n)} \in \mathcal{B}, n = 0, 1, \dots$, the sequence $\{h^{(n)}\}_{n\geqslant 0}$ is normal in $\mathbb{D}$ and therefore converges uniformly on compact subsets of $\mathbb{D}$ to f in $\mathcal{B}$ with the Taylor series at $z = 0$ equal to $\lim_n h^{(n)}$ in $\mathbb{C}[[z]]$. Passing to the limit in (23), we obtain the identity

$$f = \frac{A + zB^*\beta f}{B + zA^*\beta f}, \quad z \in \mathbb{D}, \tag{24}$$

which shows that f satisfies (22) in $\mathbb{D}$. Conditions (a) and (b) follow directly from (24). Indeed,

$$f - \frac{A}{B} = \frac{z^{d+1}\omega\beta f}{B(B + zA^*\beta f)} = O(z^{d+1}), \quad z \to 0,$$

implies (a). Then (24) implies (b). See [4, Theorem 4.6] for details. □

Remark 3.1. Theorem 3.1 can be considered as an analogue of Euler's Theorem for periodic regular continued fractions. If $\beta \equiv 1$, then by (21) the Schur parameters of f are periodic with period $d + 1$: $f = \{\overline{a_0, \ldots, a_d}\}$.

7. Let us consider the case $\beta \equiv 1$ in more details. By Theorem 3.1 every periodic Schur function is uniquely determined by an Wall pair (A, B) from the equation

$$f = \frac{A + zB^* f}{B + zA^* f}. \tag{25}$$

Equivalently, f is a solution to the quadratic equation

$$zA^*X^2 + (B - zB^*)X - A = 0. \tag{26}$$

The polynomials zA^*, $(B - zB^*)$, $-A$ are in $\mathbb{C}_{d+1}[z]$ and satisfy

$$\left(zA^*\right)^*_{d+1} = -(-A), \quad \left(B - zB^*\right)^*_{d+1} = -\left(B - zB^*\right).$$

Compare these formulas with [5, Lemma 5.2].

Lemma 3.2. *For any Wall pair (A, B) the roots of the polynomials $B - zB^*$ and $B + zB^*$ are simple, placed on $\mathbb{T}$ and interlaced.*

Proof. The roots of $B - zB^*$ are exactly the points where the Blaschke product zB^*/B assumes the value 1. Since the argument of any Blaschke product increases in the counterclockwise direction of the unit circle, this implies that the roots of $B - zB^*$ are simple, are placed on $\mathbb{T}$ and interlace the roots of $B + zB^*$. □

Theorem 3.3. *The discriminant $\mathcal{D}$, $\deg(\mathcal{D}) = 2d + 2$, of (26) satisfied by the exposed periodic quadratic irrationality determined by an Wall pair (A, B) is given by*

$$\mathcal{D} = b^2 - 4\omega z^{d+1}, \tag{27}$$

where $b = B + zB^*$ is a separable polynomial with roots on $\mathbb{T}$. The polynomial b satisfies

$$b^* = b, \quad b(0) = 1. \tag{28}$$

All roots of $\mathcal{D}$ are placed on $\mathbb{T}$ and coincide with the roots of the trigonometric polynomial $|b|^2 - 4\omega$. These roots are either simple or of the second order. The latter happens if and only if $|b|^2 - 4\omega$ has at this root a local maximum.

Proof. By (6) the discriminant $\mathcal{D}$ of (26) is

$$\mathcal{D} = \left(B - zB^*\right)^2 + 4zA^*A = \left(B + zB^*\right)^2 - 4\omega z^{d+1}, \tag{29}$$

where ω is a parameter of the Wall pair (A, B), $|B|^2 - |A|^2 = \omega$. The discriminant $\mathcal{D}$ has degree $2d + 2$ and is self-adjoint:

$$\mathcal{D}^* = z^{2d+2}\left\{\left(\overline{B\left(\frac{1}{\bar{z}}\right)} + \frac{1}{z^{d+1}}B(z)\right)^2 - 4\omega z^{-d-1}\right\} = \mathcal{D}.$$

Formula (27) follows by (29). By Lemma 3.2 the polynomial b is separable and its roots are placed on $\mathbb{T}$. Conditions (28) follow from the formula for b and the property $B(0) = 1$ of Wall polynomials. Let

$$\mathcal{T}_{\mathcal{D}}(z) = \bar{z}^{(d+1)}\mathcal{D}(z), \quad z \in \mathbb{T}. \tag{30}$$

Then

$$\mathcal{T}_{\mathcal{D}}(z) = \begin{cases} |B + zB^*|^2 - 4\omega, \\ 4|A|^2 - |B - zB^*|^2, \end{cases} \tag{31}$$

for $z \in \mathbb{T}$. Indeed, since $B + zB^*$ is self-adjoint

$$\bar{z}^{(d+1)}\mathcal{D}(z) = \bar{z}^{(d+1)}(B + zB^*)(B + zB^*)^* - 4\omega = |B + zB^*|^2 - 4\omega.$$

Since $(B - zB^*)^* = -(B - zB^*)$, we obtain that

$$\bar{z}^{(d+1)}\mathcal{D}(z) = 4|A|^2 - \bar{z}^{(d+1)}(B - zB^*)^*(B - zB^*) = 4|A|^2 - |B - zB^*|^2.$$

The first formula in (31) shows that $\mathcal{T}_{\mathcal{D}}$ takes the minimal value -4ω at $d + 1$ different zeros $\{t_j^+\}_{j=0}^d$ of $b = B + zB^*$. The second formula in (31) shows that $\mathcal{T}_{\mathcal{D}}$ takes nonnegative values at $d + 1$ different zeros $\{t_j^-\}_{j=0}^d$ of $B - zB^*$ interlacing the zeros of $B + zB^*$. It follows that on each open arc (t_j^+, t_{j+1}^+) there are either two zeros of $\mathcal{T}_{\mathcal{D}}$ or one zero of multiplicity 2. Counting zeros, we get $2d + 2$ zeros total. $\square$

Corollary 3.4. *The discriminant $\mathcal{D}$ of a periodic quadratic irrationality does not vanish in $\mathbb{D}$, and on $\mathbb{T}$ has either simple zeros or zeros of the second order.*

The polynomial $\mathcal{T}_{\mathcal{D}}$ in (30) is called the trigonometric polynomial *associated* with (A, B).

8. Choosing the sign of roots by $\sqrt{1} = 1$ and observing that $|f(0)| < 1$, we obtain an explicit formula for f

$$\begin{aligned} f(z) &= \frac{-(B - zB^*) + \sqrt{(B - zB^*)^2 + 4zAA^*}}{2zA^*} \\ &= \frac{(B - zB^*)}{2zA^*}\left\{\sqrt{1 + \frac{4zA^*A}{(B - zB^*)^2}} - 1\right\}. \end{aligned} \tag{32}$$

On the unite circle $\mathbb{T}$

$$\frac{4zA^*A}{(B-zB^*)^2} = \frac{4z^{1+d}|A|^2}{(B-zB^*)(B-zB^*)}$$
$$= -\frac{4|A|^2}{(B-zB^*)\overline{(B-zB^*)^*}} = -\frac{4|A|^2}{|B-zB^*|^2}. \tag{33}$$

Combining (32) with (33), we obtain a formula for $|f|$ on $\mathbb{T}$

$$|f| = \frac{|B-zB^*|}{2|A|}\left|1-\sqrt{1-\frac{4|A|^2}{|B-zB^*|^2}}\right|. \tag{34}$$

9. Although the coefficients of the quadratic equation (26) satisfy the conditions of Lemma 5.2 in [5], the quadratic Eq. (26) may not be irreducible over $\mathbb{C}[z]$. To see this let us consider the Wall polynomials (A_n, B_n) associated with the solution

$$f(z) = \frac{a}{1} - \frac{(1-a^2)z}{1+z} - \frac{(1-a^2)z}{1+z} - \frac{(1-a^2)z}{1+z} - \ldots \tag{35}$$

of the quadratic equation

$$azX^2 + (1-z)X - a = 0, \quad 0 < a < 1.$$

The Euler–Wallis recurrence for (35) shows that A_n and B_n satisfy the three-term recurrence equation

$$U_{n+1} = (1+z)U_n - (1-a^2)zU_{n-1}, \quad n = 1, 2, \ldots, \tag{36}$$

where

$$\begin{aligned} A_0 &= a, \quad A_1 = a + az, \\ B_0 &= 1, \quad B_1 = 1 + a^2 z. \end{aligned}$$

It is easy to see that B_n^* also satisfy (36). It follows that both polynomials are linear combinations of A_n and B_n. Elementary algebra shows that

$$B_n^* = \frac{1}{z}B_n + \frac{z-1}{az}A_n,$$

which implies

$$A_n = a\frac{B_n - zB_n^*}{1-z}.$$

Hence all roots of $A_n^* = A_n$ are located on $\mathbb{T}$ at the zeros of $B_n - zB_n^*$ and the quadratic polynomial in (26) with $A = A_n$, $B = B_n$ is not irreducible over $\mathbb{C}[z]$. By Lemma 7.5 below it, however, is irreducible over $\mathbb{C}(z)$.

4. Low extremal polynomials

10. The sign of the expression under the root in (34) is defined by the sign of $\mathcal{T}_{\mathcal{D}}$ on $\mathbb{T}$, see (31):

$$1 - \frac{4|A|^2}{|B-zB^*|^2} \quad \begin{cases} \geqslant 0 & \text{if } \mathcal{T}_{\mathcal{D}} \leqslant 0, \\ < 0 & \text{if } \mathcal{T}_{\mathcal{D}} > 0. \end{cases} \tag{37}$$

The elementary inequality $1-\sqrt{1-x^2}<x$ valid on $(0,1)$ and (34) show that

$$\begin{aligned} |f| &< 1 \text{ if } \mathcal{T}_{\mathcal{D}} < 0, \\ |f| &= 1 \text{ if } \mathcal{T}_{\mathcal{D}} \geqslant 0. \end{aligned} \tag{38}$$

The following theorem clarifies the structure of $\mathcal{T}_{\mathcal{D}}$.

Theorem 4.1. *Let $\mathcal{T}=\mathcal{T}_{\mathcal{D}}$ be the trigonometric polynomial associated with an Wall pair (A, B). Then*

(a) $\mathcal{T}(z)=\bar{z}^{d+1}+\cdots+z^{d+1},\ z\in\mathbb{T}$;
(b) $\mathcal{T}(z)$ *assumes its minimal value* $\min_{\mathbb{T}}\ \mathcal{T}(z)=-4\omega$ *at $d+1$ different points*;
(c) $\mathcal{T}(z)$ *has $d+1$ points of nonnegative local maxima*;
(d) $\mathcal{T}(z)>0$ *at least at one point of* $\mathbb{T}$.

Proof. (a) follows from the formulas $\mathcal{D}(0)=1$ and $\mathcal{D}^*=\mathcal{D}$. (b) and (c) follow from (31). By (31) $|A(t_j^-)|>0$ at least in one zero t_j^- of $B-zB^*$, since $\deg A\leqslant d$. This implies (d). $\square$

By (31) $\mathcal{T}_{\mathcal{D}}=|B+zB^*|^2-4\omega$. Thus $\mathcal{T}_{\mathcal{D}}$ is obtained from $|B+zB^*|^2$ by moving its graph down by 4ω. Theorem 4.1 shows that 4ω cannot exceed the minimal maximum of $|B+zB^*|^2$ on $\mathbb{T}$.

Definition 4.1. For a polynomial b with zeros on $\mathbb{T}$ we denote by m_b the smallest local maximum of $|b|^2$ on $\mathbb{T}$.

Definition 4.2. A real trigonometric polynomial $\mathcal{T}$ of degree $g+1$ on $\mathbb{T}$ is called lower extremal *if it satisfies conditions* (a)–(d) *of Theorem* 4.1 *with* $\omega\in(0,1)$.

This terminology is related with the extremal property of Chebyshev's polynomials $T_n(x)=\cos(n\arccos x)$ which assume consecutively maximal and minimal values ±1.

By Theorem 4.1 any Wall pair determines a lower extremal polynomial $\mathcal{T}_{\mathcal{D}}$ defined by (30). The correspondence

$$(A, B)\longrightarrow\mathcal{T}_{\mathcal{D}}$$

is not one-to-one. For example, equivalent Wall pairs correspond to one polynomial $\mathcal{T}_{\mathcal{D}}$.

Lemma 4.2. *Let $\mathcal{T}(z)$ be a lower extremal trigonometric polynomial of degree $d+1$. Then*

(a) *$\mathcal{T}(z)$ has $2(d+1)$ zeros on $\mathbb{T}$ counting the multiplicities*;
(b) *all zeros of $\mathcal{T}$ are either simple or of the second order*;
(c) *a zero of $\mathcal{T}$ is of the second order if and only if this zero is a point of local maximum for $\mathcal{T}$*;
(d) *there is a unique polynomial $b\in\mathbb{C}_{d+1}[z]$ with zeros at the points of local minima of $\mathcal{T}$ such that*

$$b(0)=1,\quad b^*=b,\quad b^2-4\omega z^{d+1}=\mathcal{D}(z)\overset{def}{=}z^{d+1}\mathcal{T}_{\mathcal{D}}(z). \tag{39}$$

Proof. The proof of (a)–(c) follows word by word the proof of Markov's Theorem [11, 3.411, p. 51].

Let $\{t_j\}_{j=0}^{d}$ be $d+1$ different points on $\mathcal{T}$ such that

$$T(t_j) = \min_{\mathbb{T}} \mathcal{T} = -4\omega, \quad \omega \in (0,1), \quad j = 0, \dots, d.$$

Then by Rolle's theorem $\dot{\mathcal{T}} = \partial \mathcal{T}/\partial\theta$ has at least one zero on each open arc (t_j, t_{j+1}), $t_{d+1} = t_0$. Since $\dot{\mathcal{T}}(t_j) = 0$, $j = 0, 1, \dots, d$, we obtain a complete list of $2(d+1)$ zeros of $\dot{\mathcal{T}}$ on $\mathbb{T}$. It follows that $\dot{\mathcal{T}}$ has exactly one zero in (t_j, t_{j+1}), $j = 0, 1, \dots, d$, and therefore $\mathcal{T}$ is monotonic between any two consecutive zeros of $\dot{\mathcal{T}}$. Since $\mathcal{T}$ is lower extremal, any open arc (t_j, t_{j+1}) contains either two different zeros of $\mathcal{T}$ or one zero of multiplicity two. This yields (a)–(c).

To prove (d) we consider a nonnegative trigonometric polynomial $\mathcal{T}(z) + 4\omega$ on $\mathbb{T}$, which by Fejer's theorem [11, Theorem 1.2.2] can be uniquely represented as

$$\mathcal{T}(z) + 4\omega = |b|^2, \quad b \in \mathbb{C}_{d+1}[z], \quad b(0) > 0. \tag{40}$$

Hence

$$bb^* - 4\omega z^{d+1} = z^{d+1}\mathcal{T}(z) = \mathcal{D}(z) \in \mathbb{C}_{2(d+1)}[z].$$

Comparing the constant terms of bb^* and $\mathcal{D}$, we see that $b_{d+1}\bar{b}_0 = 1$, where

$$b(z) = b_0 z^{g+1} + \cdots + b_{d+1}, \quad b_{d+1} = b(0) > 0.$$

On the other hand, $b^* = \lambda b$, $\lambda \in \mathbb{T}$, since by (40) all zeros of b lie on $\mathbb{T}$. Hence $\bar{b}_0 = \lambda b_{d+1}$ and therefore $1 = b_{d+1}\bar{b}_0 = \lambda b_{d+1}^2$. It follows that $b_{d+1} = b(0) = 1$, $\lambda = 1$. $\square$

Remark 4.1. By (4) of Lemma 4.2 any lower extremal trigonometric polynomial has the form $|b|^2 - 4\omega$, where b is a polynomial in z with all roots on $\mathbb{T}$, $b^* = b$, $b(0) = 1$, and $4\omega \leqslant m_b$.

Remark 4.2. By Lemma 4.2 the discriminant $\mathcal{D}(z) = z^{d+1}\mathcal{T}(z)$ can be uniquely factored as $\mathcal{D} = p^2\mathcal{D}_0$, where p and $\mathcal{D}_0$ are separable polynomials. Then by (39) b and p satisfy *Pell's* equation

$$b^2 - p^2\mathcal{D}_0 = 4\omega z^{d+1}.$$

In the approach developed by Peherstoffer and Steinbauer for a description of measures with periodic Verblunsky parameters the polynomial b is called a T-polynomial, see [7, (3.12)].

If b is separable, $\deg(b) = d+1$, $4\omega < m_b$, then the set $\{t \in \mathbb{T} : |b|^2 - 4\omega < 0\}$ consists of $d+1$ open arcs separated by closed arcs not reducing to a point. When $\omega \uparrow m_b/4$ some of these intervals touch each other at the points of the local maximum m_b. This observation allows one to consider this very case as a limit one.

5. Applications

11. By Theorem 4.1 (d) and (38) $|f| = 1$ at least on a small arc of $\mathbb{T}$. Herglotz' formula

$$\frac{1+zf}{1-zf} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) \tag{41}$$

shows that the negative sign of $\mathcal{T}_{\mathcal{D}}$ controls the essential support $\text{supp}_e(\sigma)$ for the representing measure σ of *f*. In what follows we sometime write $\sigma = \sigma_f$ for the representing measure σ for *f*.

Lemma 5.1. *If f is a nonzero periodic Schur function, then* $|f| = 1$ *on a nonempty arc of* $\mathbb{T}$.

Proof. By Theorem 4.1 (d) $\mathcal{T}_{\mathcal{D}} > 0$ on an nonempty arc of $\mathbb{T}$. The statement of the Lemma now follows from (38). $\square$

Since $|f| = 1$ at least on some open nonempty arc, we obtain that $\text{supp}_e(\sigma) \neq \mathbb{T}$. The Lebesgue decomposition $d\sigma = \sigma'\,dm + d\mu_s$ into the absolutely continuous part $\sigma'\,dm$ and the singular part $d\mu_s$ has some peculiarities. First of all by Fatou's Theorem

$$\sigma' = \frac{d\sigma}{dm} = \frac{1-|f|^2}{|1-zf|^2}.$$

Observing that an algebraic function *f* is continuous on $\mathbb{T}$, see for instance [5, Lemma 2.4], we obtain that $\sigma' > 0$ on a finite system of open arcs $\delta_j = (\tau_j^-, \tau_j^+)$ interlaced with closed arcs $\gamma_j = [\tau_j^+, \tau_{j+1}^-]$, $j = 0, 1, \dots, d$. Symbol "−" indicates that $\mathcal{T}_{\mathcal{D}}$ passes from positive values to negative in the anti-counterclockwise direction on $\mathbb{T}$. Then

$$\mathcal{E}(f) = \bigcup_{j=0}^{d} \delta_j, \quad \mathcal{U}(f) = \bigcup_{j=0}^{d} \gamma_j.$$

Thus, the *essential support* of σ (of *f*) is $\mathcal{E}(f)$. Notice that $\mathcal{E}(f)$ is an open subset of $\mathbb{T}$, whereas $\mathcal{U}(f)$ is closed. Measure σ may have a discrete spectrum. It is located on $\mathbb{T}$ at the roots of an algebraic equation $zf - 1 = 0$. It follows that the number of point masses is finite. Since $|f| = 1$ if $zf = 1$, these masses are located in $\mathcal{U}(f)$. It is important to notice that closed arcs γ_j may reduce to one point.

Theorem 5.2. *Given a pair f, g of periodic Schur functions* $\mathcal{E}(f) = \mathcal{E}(g)$ *if and only if f and g have equal discriminants.*

Proof. By (30) periodic Schur functions with equal discriminants have a common lower extremal polynomial $\mathcal{T}$, which implies

$$\mathcal{E}(f) = \mathcal{E}(g) = \{t \in \mathbb{T} : \mathcal{T} < 0\}. \tag{42}$$

Conversely, if (42) holds, then by Theorem 4.1 (a)

$$|b_f|^2 - 4\omega_f = |b_g|^2 - 4\omega_g,$$

since the zero sets of both trigonometrical polynomials coincide. This implies by (30) that the discriminants are equal as well. $\square$

Theorem 5.3. *Every closed arc* γ_j, $j = 0, 1, \dots d$, *contains at most one point mass of the representative measure* σ *for a periodic Schur function f.*

Proof. If $x_1 = f$, x_2 are the roots of (26), then $zx_1 - 1$ and $zx_2 - 1$ by Vieta's Theorem satisfy

$$A^*(zx_1 - 1)(zx_2 - 1) = B + A^* - z(A + B^*). \tag{43}$$

Since $A/B \in \mathcal{B}$ and $|A| = |A^*|$ on $\mathbb{T}$, the polynomial $B + A^* = B(1 + A^*/B)$ does not vanish in Clos $\mathbb{D}$. Therefore, the roots of the adjoint polynomial $(B + A^*)^* = A + B^*$ must be in $\mathbb{D}$. It follows that the roots of $zf - 1 = 0$ lie in the set

$$1 = z\frac{(B + A^*)^*}{B + A^*} = z\frac{B^*}{B}\frac{\overline{h}}{h}. \tag{44}$$

Here $h = 1 + A^*/B$ is a smooth outer function in Clos $\mathbb{D}$ with values in the *open* right half-plane. Let us observe that the function in the middle of (44) is a finite Blaschke product. Since its argument increases when t moves anti-counterclockwise, (44) has exactly $d + 1$ solutions. Since $\mathcal{T}_{\mathcal{D}} = |B + zB^*|^2 - 4\omega$, the roots of

$$-1 = z\frac{B^*}{B} \tag{45}$$

are located inside open intervals δ_j so that each interval δ_j has exactly one root t_j^+. On $[t_j^+, t_{j+1}^+]$ the argument $\alpha(t)$ of the Blaschke product zB^*/B increases from $-\pi$ to $+\pi$. For h we have

$$h = |h| \exp i\beta t, \quad -\frac{\pi}{2} < \beta(t) < \frac{\pi}{2}, \quad \frac{\overline{h}}{h} = \exp -2i\beta(t). \tag{46}$$

It follows that a continuous function $u(t) = \alpha(t) - 2\beta(t)$ assumes the values of opposite sign on $[t_j^+, t_{j+1}^+]$. Therefore, it must vanish inside of $[t_j^+, t_{j+1}^+]$ and by (44) the right-hand side of (43) vanishes. In the open interval (t_j^+, t_{j+1}^+) there is exactly one closed arc of $\mathcal{U}(f)$, namely γ_j. Therefore, every arc γ_j contains at most one root of $B + A^* - z(A + B^*) = 0$. Since the number of roots equals the number of arcs the result follows. $\square$

Remark 5.1. The result on the location of point masses was obtained in a preliminary form by Geronimus in [3]. In the form it is stated here, the result was obtained by Simon in [10] (see also [9]). The proof given here is different from those given in [3,10].

12. Discrete masses of σ are located at the roots of $1 - zf = 0$ of order 1. Since $\Re(1 - zf) \geqslant 0$, this order cannot exceed 1. The end-points of the essential support, where the discriminant has simple zeros, are branching points of order 2 and therefore the order of the zero is 1/2. This implies that $(1 - zf)^{-1}$ is locally integrable at these points. The following theorem gives an explicit formula for the density of the periodic measure associated with an Wall pair.

Theorem 5.4. *Let σ be the periodic measure with Schur function f associated with an Wall pair (A, B) of degree d. Then*

$$\sigma' = \frac{\sqrt{\mathcal{D}}}{B + A^* - z(B^* + A)} = \frac{\sqrt{\mathcal{D}}}{\Phi_{d+1}^* - \Phi_{d+1}} = \frac{\sqrt{|\mathcal{D}|}}{|\Phi_{d+1}^* - \Phi_{d+1}|}, \tag{47}$$

where Φ_{d+1} is the monic orthogonal polynomial of order $d + 1$ and the formula holds on $\mathcal{E}(f)$.

Proof. We denote by u the rational function $2A^*/(B - zB^*)$. Then by (34)

$$zf = -\frac{1 - \sqrt{1 - |u|^2}}{u} \tag{48}$$

on $\mathbb{T}$. It follows that

$$1-|f|^2=\frac{|u|^2-(1-\sqrt{1-|u|^2})^2}{|u|^2}$$
$$=\frac{|u|^2-1-(1-|u|^2)+2\sqrt{1-|u|^2}}{|u|^2}=2\sqrt{1-|u|^2}\,\frac{1-\sqrt{1-|u|^2}}{|u|^2}. \tag{49}$$

Next,

$$\frac{1-|f|^2}{|1-zf|^2}=\frac{1-|f|^2}{1+|f|^2-2\Re(zf)}=\frac{1}{2\dfrac{1-\Re(zf)}{1-|f|^2}-1}.$$

Using (48) and (49), we obtain

$$2\,\frac{1-\Re(zf)}{1-|f|^2}-1=\frac{1+\left(1-\sqrt{1-|u|^2}\right)\Re\left(\frac{1}{u}\right)}{\sqrt{1-|u|^2}\,\dfrac{1-\sqrt{1-|u|^2}}{|u|^2}}-1$$
$$=\frac{\dfrac{|u|^2}{1-\sqrt{1-|u|^2}}+|u|^2\Re\left(\dfrac{1}{u}\right)}{\sqrt{1-|u|^2}}-1=\frac{1+\Re(u)}{\sqrt{1-|u|^2}}. \tag{50}$$

To find $\Re(u)$ we observe that

$$\bar{u}=\frac{2\overline{A^*}}{\overline{B-zB^*}}=\frac{2\bar{z}^dA}{\overline{B}-\bar{z}^{(1+d)}B}=-\frac{2zA}{B-zB^*}, \tag{51}$$

which implies

$$\Re(u)=\frac{1}{2}(u+\bar{u})=\frac{A^*-zA}{B-zB^*}.$$

Hence

$$1+\Re(u)=\frac{B+A^*-z(B^*-A)}{B-zB^*}. \tag{52}$$

Combining (50) with (52), we obtain the required formula. The last formula with orthogonal polynomials follows from the known relationship between orthogonal polynomials and Wall polynomials (see [4, (5.5)]). □

Remark 5.2. A formula for σ' was first obtained by Geronimus [3] in a similar form. However, the formula for the denominator of the density was not so simple. Later Geronimus' formula was used by Peherstorfer and Steinbauer [7] to develop the Theory of Periodic Measures on the unit circle. However, the formula with $\Phi^*_{d+1}-\Phi_{d+1}$ was obtained only recently by Simon [10].

Theorem 5.5. *Let σ be the periodic measure with Schur function f associated with an Wall pair (A, B) of degree d. Then*

$$\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\sigma(\zeta)=\frac{1+zf}{1-zf}$$

$$= \frac{A^* + zA + \sqrt{\mathcal{D}}}{B + A^* - z(B^* + A)} = \frac{A^* + zA + \sqrt{\mathcal{D}}}{\Phi^*_{d+1} - \Phi_{d+1}}.$$

Proof. Let $x_1 = f$ and x_2 be two roots of (26). Then by Vieta's formulas we have

$$z(x_1 + x_2) = -\frac{B - zB^*}{A^*}, \quad z(x_1 - x_2) = \frac{\sqrt{\mathcal{D}}}{A^*}, \quad z^2 x_1 x_2 = -\frac{A}{A^*}z.$$

It follows that

$$\begin{aligned} \frac{1+zf}{1-zf} &= \frac{(1+zx_1)(1-zx_2)}{(1-zx_1)(1-zx_2)} \\ &= \frac{1 + z(x_1 - x_2) - z^2 x_1 x_2}{1 - z(x_1 - x_2) + z^2 x_1 x_2} = \frac{A^* + zA + \sqrt{\mathcal{D}}}{(B + A^*) - z(B^* + A)} \end{aligned}$$

as stated. □

6. Examples of periodic Schur functions

By (31) all periodic Schur functions with fixed polynomial $b_+(z) = B + zB^*$ and ω have the same essential support $\mathcal{E} = \{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} \leqslant 0\}$. If a set $\mathcal{E}$ essentially supports a periodic Schur function f, then there are other periodic Schur functions with the same support. The first example of such a function is given by the Wall pair $(-A^*, B)$:

$$f^\# = \frac{-A^* + zB^* f^\#}{B - zAf^\#}.$$

Function $f^\#$ satisfies the quadratic equation

$$zAX^2 - (B - zB^*)X - A^* = 0.$$

If $\{a_0, \ldots, a_d\}$ is a period of f, then $\{-\bar{a}_d, \ldots, -\bar{a}_0\}$ is the period of $f^\#$.

More examples of Schur functions with the same essential support can be obtained by the cyclic shift of parameters:

$$\begin{array}{lcllcll} f_1 & = & \{a_1, & a_2, & \ldots, & a_d, & a_0\} \\ f_2 & = & \{a_2, & a_3, & \ldots, & a_0, & a_1\} \\ \vdots & & & & & & \\ f_d & = & \{a_d, & a_0, & \ldots, & a_{d-2}, & a_{d-1}\}. \end{array} \tag{53}$$

The formula

$$1 - |f_n|^2 = \frac{(1 - |a_n|^2)(1 - |f_{n+1}|^2)}{|1 + \bar{a}_n z f_{n+1}|^2}, \quad z \in \mathbb{T},$$

shows that $\mathcal{U}(f) = \mathcal{U}(f_n)$, which implies that all periodic functions in (53) have the same essential support. Hence by Theorem 5.2 they have a common discriminant. By [4, Theorem 4.9, p. 195]

$$B + zA^* f = \prod_{k=0}^{d} (1 + z\bar{a}_k f_{k+1}). \tag{54}$$

It follows that the combination $B + zA^* f$ remains invariant under the cyclic shift. Moreover, this expression is one and the same for any exposed periodic irrationality with given discriminant $\mathcal{D}$. Indeed, by (32)

$$\begin{aligned}\rho_1(z) &= \frac{B + zA^* f}{\sqrt{\omega}} = \frac{1}{2\sqrt{\omega}} \left\{(B + zB^*) + \sqrt{\mathcal{D}}\right\} \\ &= \frac{B + zB^*}{2\sqrt{\omega}} \left\{1 + \sqrt{1 - \frac{4\omega z^{d+1}}{(B + zB^*)^2}}\right\}.\end{aligned} \tag{55}$$

Since $\mathcal{D} = (B + zB^*)^2 - 4\omega z^{d+1}$ by (29), the result follows. Hence all periodic Schur's functions belonging to one discriminant $\mathcal{D} = b_+^2 - 4\omega z^{1+d}$ are essentially supported by the same system of arcs. In other words algebraic function $\rho_1(z)$ controls the corresponding set of periodic Schur functions.

Now for $z \in \mathbb{T}$

$$\frac{4\omega z^{d+1}}{(B + zB^*)^2} = \frac{4\omega}{(B + zB^*)\bar{z}^{d+1}(B + zB^*)} = \frac{4\omega}{|B + zB^*|^2}.$$

This shows that

$$\frac{4\omega z^{d+1}}{(B + zB^*)^2}$$

is positive on $\mathbb{T}$. Taking (31) and (55) into account, we obtain that

$$\begin{aligned}&|\rho_1| = 1 \quad \text{if } \mathcal{T}_{\mathcal{D}} < 0, \\ &|\rho_1| \geqslant 1 \quad \text{if } \mathcal{T}_{\mathcal{D}} \geqslant 0.\end{aligned} \tag{56}$$

7. Galois theorem for Schur's algorithm

13. A quadratic irrationality ξ (over $\mathbb{Q}$) is called *reduced* if $\xi > 1$ and the algebraic conjugate irrationality ξ' belongs to the open interval $(-1, 0)$. Purely periodic real quadratic irrationalities are characterized by Galois' theorem.

Theorem 7.1 (*Galois, [1]*). *A regular continued fraction is purely periodic if and only if its value is a reduced quadratic irrationality.*

See the proof in [6]. Theorem 7.1 was the first published result of Galois, who seemingly attempted to apply continued fractions to solve the Basic Problem of Algebra on algebraic equations.

In this section we consider an extension of Galois' Theorem to Schur's Algorithm. Multiplying (24) by β, we obtain that $h = \beta f \in \mathcal{B}$ satisfies

$$h = \beta \frac{A + zB^* h}{B + zA^* h},$$

which, in fact, is equivalent to (24). Indeed, since (A, B) is an Wall pair, the fraction in the above formula must lie in $\mathcal{B}$. Hence $f = h\beta^{-1} \in \mathcal{B}$.

Applying Theorem 3.1 to two equivalent Wall pairs (A, B) and $(-A^*, B)$ with a fixed function $\beta \in \mathcal{B}$, we obtain two functions f and $f^\#$ in $\mathcal{B}$ satisfying

$$f = \frac{A + zB^*\beta f}{B + zA^*\beta f}, \quad f^\# = \beta\frac{-A^* + zB^* f^\#}{B - zAf^\#}, \quad z \in \mathbb{D}. \tag{57}$$

Corollary 7.2. *The roots of* (22) *in* $\mathbb{D}$ *are given by*

$$x_1 = f, \quad x_2 = \frac{1}{zf^\#}.$$

The discriminant of (22) *does not vanish in* $\mathbb{D}$.

Proof. The second formula of (57) implies that $f^\#$ satisfies

$$zAX^2 - (B - zB^*\beta)X - A^*\beta = 0 \tag{58}$$

and

$$A - (B - zB^*\beta)\frac{1}{zf^\#} - zA^*\beta\left(\frac{1}{zf^\#}\right)^2 = 0.$$

It follows that $(zf^\#)^{-1}$ satisfies (22) in $\mathbb{D}$. Since obviously $|(zf^\#)^{-1}| > 1$ in $\mathbb{D}$, we conclude that $(zf^\#)^{-1} = x_2$ and that

$$x_1 - x_2 = \frac{zff^\# - 1}{zf^\#} \neq 0 \quad \text{in } \mathbb{D}$$

Since $z\beta B^* B^{-1} \in \mathcal{B}$, we see that $(B - zB^*\beta) = B(1 - z\beta B^* B^{-1})$ cannot vanish in $\mathbb{D}$, which implies that the discriminant of (22)

$$\mathcal{D}(z) = (B - zB^*\beta)^2 + 4z\beta A^* A = (x_1 - x_2)^2(zA^*\beta)^2$$

cannot vanish at the zeros of $zA^*\beta$ in $\mathbb{D}$. Since $x_1 \neq x_2$ in $\mathbb{D}$, the proof is completed. □

Similarly, we obtain that the roots y_1, y_2 of (58) in $\mathbb{D}$ are given by

$$y_1 = f^\#, \quad y_2 = \frac{1}{zf}.$$

Notice that the discriminants of (58) and (22) are identical.

14. It is clear that Eqs. (22) and (58), as well as their roots, are related by a duality.

Definition 7.1. The function $f^\#$ is called the Galois dual function for f (with respect to (A, B) and β).

The Vieta Theorem for (22) implies

$$f^\# = -\frac{A^*}{A}\beta f. \tag{59}$$

It follows that $|f^\#| = |\beta f|$ on $\mathbb{T}$ and, in particular, $|f| = |f^\#|$ on $\mathbb{T}$, if β is an inner function. The terminology of Galois dual functions is related with a well-known theorem due to Galois.

Theorem 7.3 (*Galois, [1]*). *Let α be an algebraic number with periodic regular continued fraction $\alpha = [\overline{b_0, \ldots, b_d}]$, $b_j \in \mathbb{N}$, $j = 0, \ldots, d$, τ an automorphism in Gal$(\mathbb{Q}(\alpha)/\mathbb{Q})$ with $\tau\alpha \neq \alpha$. Then*

$$-\frac{1}{\tau\alpha} = [\overline{b_d, b_{d-1}, \ldots, b_0}].$$

An analogue of this theorem for Wall periodic continued fractions looks as follows.

Theorem 7.4. *Let $f = \{\overline{a_0, \ldots, a_d}\}$ be a Schur function in $\mathcal{B}$ with periodic Schur parameters with $\max_j |a_j| > 0$, w the algebraic function corresponding to f, τ an automorphism in Gal$(\mathbb{C}(z, w)/\mathbb{C}(z))$ with $\tau f \neq f$. Then*

$$-\frac{1}{z\tau f} = \{\overline{\bar{a}_d, \bar{a}_{d-1}, \ldots, \bar{a}_0}\} = -f^{\#}.$$

Proof. By [4, Lemma 5.13] we have

$$-b_{n+1} = \frac{A_n^*}{B_n} - \frac{\omega_n z^{n+1}}{B_n(B_n - zA_n)}.$$

Combining this formula with

$$b_{n+1}(z) = \frac{zb_n(z) - \bar{a}_n}{1 - za_n b_n(z)}$$

(see [4, (7.11)]), we obtain

$$\mathcal{S}\left(\frac{A_n^*}{B_n}\right) = (\bar{a}_n, \bar{a}_{n-1}, \ldots, \bar{a}_0, 0, 0, \ldots).$$

If we put here $n = d$, $A_n = A$, $B_n = B$, we obtain that the Wall pair (A^*, B) corresponds to the parameters $\bar{a}_d, \bar{a}_{d-1}, \ldots, \bar{a}_0$ and by Theorem 3.1 to the equation

$$zAX^2 + (B - zB^*)X - A^* = 0,$$

which is satisfied by $-f^{\#}$, $f^{\#} \in \mathcal{B}$, see (58) with $\beta \equiv 1$. By (8) $-f^{\#}$ is periodic and

$$-f^{\#} = \{\overline{\bar{a}_d, \bar{a}_{d-1}, \ldots, \bar{a}_0}\}.$$

Since $\tau f \neq f$, we must have $\tau f = x_2 = 1/zf^{\#}$ by Corollary 7.2. $\square$

Lemma 7.5. *The quadratic equation for $f = \{\overline{a_0, \ldots, a_d}\}$ with* $\max |a_j| > 0$ *is irreducible over* $\mathbb{C}(z)$

Proof. Since $\max |a_j| > 0$, the parameters of f do not terminate, which implies that f is not a finite Blaschke product. It follows that $m(\mathcal{U}(f)) < 1$. By Lemma 5.1 we have $0 < m(\mathcal{U}(f)) < 1$. Hence, if the algebraic equation for f is reducible over the field of rational functions, then f is a rational function itself. Then $f^*(z) = \overline{f(1/\bar{z})}$ is also rational implying that ff^* is rational as well. But $ff^* = 1$ on $\mathcal{U}(f)$, which is a proper subset of $\mathbb{T}$, contradicting to the fact that ff^* is rational. $\square$

8. Measures with Schur functions $ff^{\#}$

15. By Corollary 7.2 $x_1/x_2 = zff^{\#} \in \mathcal{B}$. Since by Vieta's Theorem

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{(x_1+x_2)^2}{x_1 x_2} - 2 = -\left\{2 + \frac{(B - zB^*\beta)^2}{zA^*A\beta}\right\},$$

the Schur function $ff^{\#}$ satisfies the quadratic equation

$$z^2 AA^*\beta X^2 + \{(B - zB^*)^2 + 2zAA^*\beta\}X + AA^*\beta = 0.$$

The discriminant of the quadratic equation for $ff^{\#}$ is

$$\left((B - zB^*\beta)^2 + 2zAA^*\beta\right)^2 - 4(zAA^*\beta)^2 = (B - zB^*\beta)^2\mathcal{D},$$

where $\mathcal{D} = (B - zB^*\beta)^2 + 4zA^*A\beta$ is the discriminant of the quadratic equation for f.

An important Schur function $ff^{\#}$ corresponds to the probability measure $\sigma_{ff^{\#}}$ on $\mathbb{T}$, which can be written down explicitly. In what follows we consider two auxiliary functions

$$\Omega(z) = 1 + \frac{4zA^*A\beta}{(B - zB^*\beta)^2}, \quad \mathcal{R}(z) = 1 - \frac{4\omega z^{d+1}\beta}{(B + zB^*\beta)^2}. \tag{60}$$

Since the Wall polynomial B does not vanish in $\mathrm{Clos}(\mathbb{D})$ and $\beta \in \mathcal{B}$, we see that both functions in (60) are holomorphic in $\mathbb{D}$ and satisfy $\Omega(0) = \mathcal{R}(0) = 1$. A direct calculation with the determinant identity shows that Ω and $\mathcal{R}$ are related by the following formula:

$$\Omega(z) = \left(\frac{B + zB^*\beta}{B - zB^*\beta}\right)^2 \mathcal{R}(z). \tag{61}$$

Theorem 8.1. *Let $\sqrt{w}$ be the main branch of the square root defined by $\sqrt{1} = 1$ in $\mathbb{C} \setminus (-\infty, 0)$. Then $\Re\sqrt{\Omega} > 0$ in $\mathbb{D}$ and*

$$\frac{B - zB^*\beta}{\sqrt{\mathcal{D}}} = \frac{1}{\sqrt{\Omega}} = \frac{1 + zff^{\#}}{1 - zff^{\#}} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma_{ff^{\#}}(\zeta). \tag{62}$$

Proof. Resolving (21) at a neighborhood of $z = 0$, we obtain an explicit formula for f:

$$f(z) = \frac{B - zB^*\beta}{2zA^*\beta}\left\{\sqrt{\Omega} - 1\right\} = \frac{2A}{B - zB^*\beta}\frac{1}{1 + \sqrt{\Omega}}. \tag{63}$$

Similarly, resolving (58), we have

$$f^{\#}(z) = \frac{B - zB^*\beta}{2zA}\left\{1 - \sqrt{\Omega}\right\} = -\frac{2A^*\beta}{B - zB^*\beta}\frac{1}{1 + \sqrt{\Omega}},$$

which follows from (63) by (59). Hence

$$zff^{\#} = \frac{1 - \Omega}{(1 + \sqrt{\Omega})^2} = \frac{1 - \sqrt{\Omega}}{1 + \sqrt{\Omega}},$$

which implies the validity of (62) at least at a neighborhood of $z = 0$. Now, since by Theorem 3.1 both f and $f^{\#}$ lie in $\mathcal{B}$, we obtain that (62) holds in $\mathbb{D}$ by the uniqueness theorem for holomorphic functions. Finally,

$$\Re \frac{1}{\sqrt{\Omega(z)}} = \frac{1 - |zff^{\#}|^2}{|1 - zff^{\#}|^2}, \quad z \in \mathbb{D},$$

and therefore $\Re\sqrt{\Omega} > 0$ in $\mathbb{D}$ as stated. □

Notice that $\Omega : \mathbb{D} \to \mathbb{C} \setminus (-\infty, 0]$, since $\Re\sqrt{\Omega} > 0$ in $\mathbb{D}$. Next, by Theorem 8.1 we obtain from (61) that

$$\frac{1}{\sqrt{\Omega}} = \frac{1 - zB^*\beta/B}{1 + zB^*\beta/B} \cdot \frac{1}{\sqrt{\mathcal{R}(z)}}. \tag{64}$$

It follows from (60) that $\mathcal{R}(z) = 1 + O(z^{d+1})$ as $z \to 0$. Therefore, the first d parameters of $-\beta B^* B^{-1}$ and of $ff^{\#}$ coincide.

16. Let us consider a particular interesting case $\beta \equiv 1$. Then $-B^*/B$ is a finite Blaschke product and it is the Schur function of a discrete probability measure μ_B placed at the zeros of $B + zB^*$.

Corollary 8.2. *For every pair of polynomials p, q such that the spectrum of the trigonometric polynomial $p\,\overline{q}$ lies in $[-d, d]$ we have*

$$\int_{\mathbb{T}} p\,\overline{q}\,d\sigma_{ff^{\#}} = \int_{\mathbb{T}} p\,\overline{q}\,d\mu_B. \tag{65}$$

In particular, measures μ_B and $\sigma_{ff^{\#}}$ have the same orthogonal polynomials of order not exceeding d.

Proof. It follows from (64) that

$$F_{\sigma_{ff^{\#}}}(z) = F_{\mu_B}(z) + O(z^{d+1}) \quad \text{as } z \longrightarrow 0,$$

which says that the Fourier coefficients of both measures coincide up to the index d. This is equivalent to (65). □

Corollary 8.3. *Let f be the periodic Schur function corresponding to an Wall pair (A, B). Then*

$$\int_{\mathbb{T}} (B + \zeta B^*)\overline{\zeta}^k\, d\sigma_{ff^{\#}}(\zeta) = 0, \quad k = 1, 2, \ldots, d.$$

Proof. Apply (65) to $p = B + zB^*$, $q = \zeta^k$ and observe that $B + zB^*$ vanishes on μ_B. □

Corollary 8.4. *Let $f = \{\overline{a_0, \ldots, a_d}\}$ with $\max|a_j| > 0$, σ be the probability measure with $\mathcal{H}(\sigma) = f$, $\{\varphi_n\}_{n\geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma)$. Then $ff^{\#}$ is the Schur function of the limit measure*

$$d\sigma^{(0)} = (*) - \lim_{n\equiv 0 \pmod{d+1}} |\varphi_n|^2 d\sigma.$$

Proof. By Theorem 7.4

$$b_n = f^{\#} + O(z^{n+1}), \quad z \to 0,$$

if $n \equiv 0(\bmod\, d+1)$. On the other hand, $f_n \equiv f$ for such n's in view of the periodicity of the Schur parameters of *f*. Now the result follows by [4, Theorem 3, p. 174]. □

17. Let now $f \in \mathcal{B}$ satisfy (26) and $f^{\#} \in \mathcal{B}$ satisfy

$$zAX^2 - (B - zB^*)X - A^* = 0.$$

Then (59) shows that $f^{\#}$ and $ff^{\#}$ are Schur functions essentially supported by $\mathcal{E}(f)$.

Theorem 8.5. *For every periodic f the probability measure $\sigma_{ff^{\#}}$ is absolutely continuous, is supported by $\mathcal{E}(f)$ and*

$$\int_{\mathcal{E}(f)} \frac{\zeta + z}{\zeta - z}\, d\sigma_{ff^{\#}}(\zeta) = \frac{B - zB^*}{\sqrt{D}}, \tag{66}$$

$$(\sigma_{ff^{\#}})' = \frac{B - zB^*}{\sqrt{D}} \quad \text{on } \mathcal{E}(f). \tag{67}$$

Proof. We have

$$1 - zff^{\#} = -\frac{\sqrt{\mathcal{D}}}{zA^*x_2}$$

and

$$1 + zff^{\#} = 1 + \frac{x_1}{x_2} = -\frac{B - zB^*}{zA^*x_2},$$

which obviously imply (66). This shows that $\sigma_{ff^{\#}}$ is absolutely continuous about simple zeros of $\mathcal{D}$.

The first formula in (31) shows that multiple zeros may occur at the points of the local maximum of $|B + zB^*|^2 - 4\omega$ in case this maximum is zero. Elementary counting of the zeros of the arc derivative of $\mathcal{T}_{\mathcal{D}}$ shows that all its zeros are simple. Since the orders of the corresponding zeros of $\mathcal{D}$ and $\mathcal{T}_{\mathcal{D}}$ coincide, $\mathcal{D}$ may have multiple zeros of order 2 only. Let $t_0 \in \mathbb{T}$ be the zero of the second order for $\mathcal{D}$. Then by the above arguments $\mathcal{T}_{\mathcal{D}} < 0$ about t. It follows that t_0 is surrounded by two consecutive zeros of $B + zB^*$ where $\mathcal{T}_{\mathcal{D}}$ assumes its minimal value -4ω. On the other hand, zeros of $B - zB^*$ interlace zeros of $B + zB^*$. To see this consider the Blaschke product zB^*/B and its level sets corresponding to $+1$ and -1. Since $B - zB^*$ cannot vanish on the set $\mathcal{T}_{\mathcal{D}}$ (see the first formula of (31)), we obtain $(B - zB^*)(t_0) = 0$ implying that the right-hand side of (66) cannot have a pole at t_0.

By (33)

$$\frac{\mathcal{D}}{(B - zB^*)^2} = 1 - \frac{4|A|^2}{|B - zB^*|^2},$$

which is nonnegative on $\mathcal{E}(f)$ and nonpositive on $\mathcal{U}$ by (37). Fatou's Theorem and (66) imply (67). □

Corollary 8.6. *For any Wall pair* (A, B)

$$\frac{B - zB^*}{\sqrt{D}}$$

is positive on $\mathcal{E}(f)$ *and is pure imaginary on* $\mathcal{U}(f)$

Corollary 8.7.

$$\frac{B - zB^*}{\sqrt{\mathcal{D}}} \geqslant 1 \quad on\ \{t \in \mathbb{T} : \mathcal{T} < 0\}.$$

Proof. By (31) and (67)

$$\frac{B - zB^*}{\sqrt{\mathcal{D}}} = \left|\frac{B - zB^*}{\sqrt{\mathcal{D}}}\right| = \frac{|B - zB^*|}{\sqrt{|B - zB^*|^2 - 4|A|^2}} \geqslant 1$$

on $\mathcal{E}(f)$. $\square$

18. Our next goal is to describe polynomials $r^* = -r$, $r(0) = 1$, $\deg(r) \leqslant d + 1$ such that $r/\sqrt{\mathcal{D}} \geqslant 1$ on $\{t \in \mathbb{T} : \mathcal{T} < 0\}$.

Theorem 8.8. *Let* P, Q *be polynomials such that* $\deg(P) = \deg(Q) = d + 1$, Q *a separable polynomial*, *the roots of* P *and* Q *be placed on* $\mathbb{T}$, *and*

$$P^* = -P, \quad Q^* = Q, \quad P(0) = Q(0) = 1. \tag{68}$$

Then

$$\frac{P(z)}{Q(z)} = \sum_{j=0}^{d} \frac{z_j + z}{z_j - z}\delta_j \tag{69}$$

with real δ_j.

Proof. Formula (69) is a decomposition of P/Q into partial fractions so that

$$2\delta_j = -\bar{z}_j \frac{P(z_j)}{Q'(z_j)}, \quad j = 0, 1, \ldots, d.$$

Next,

$$\overset{\bullet}{Q} = \frac{\partial Q}{\partial \theta} = Q'(z) \cdot iz$$

shows that

$$2\delta_j = \frac{1}{i} \cdot \frac{P(z_j)}{\overset{\bullet}{Q}(z_j)}.$$

By (68) on $\mathbb{T}$

$$\bar{z}^{\frac{d+1}{2}} Q = z^{\frac{d+1}{2}} \overline{Q}, \quad \bar{z}^{\frac{d+1}{2}} P = -z^{\frac{d+1}{2}} \overline{P}.$$

It follows that

$$\varphi(z) = \bar{z}^{\frac{d+1}{2}} Q(z), \quad \psi(z) = i\bar{z}^{\frac{d+1}{2}} P(z), \ \ z \in \mathbb{T}$$

are real functions, which implies, since $Q(z_j) = 0$, that

$$\overset{\bullet}{\varphi}(z_j) = \bar{z}_j^{\frac{d+1}{2}} \overset{\bullet}{Q}(z_j)$$

and $\psi(z_j)$ are real numbers. It follows that all δ_j are real and

$$\sum_{j=0}^{d} \delta_j = \frac{P(0)}{Q(0)} = 1. \quad \square \tag{70}$$

Corollary 8.9. *Let P and Q be separable polynomials of degree $d+1$ with alternating zeros on $\mathbb{T}$ satisfying* (68). *Then all δ_j in* (70) *are positive.*

Proof. Since all roots of Q are simple, the roots of φ are simple too. This implies that the signs of the derivatives $\overset{\bullet}{\varphi}(z_j)$ alternate. Hence in order the numbers δ_j be positive, it is necessary and sufficient that $\psi(z_j)$ be alternating. $\square$

Definition 8.1. A pair (P, Q) of polynomials satisfying the conditions of Corollary 8.9 is called alternating.

Corollary 8.10. *A pair of polynomials (P, Q) is alternating if and only if there exists a polynomial B, $B(0) = 1$, not vanishing in Clos $\mathbb{D}$, such that*

$$P = B - zB^*, \quad Q = B + zB^*.$$

Proof. By Corollary 8.9

$$0 < 1 \leqslant 1 + \Re\left(\frac{P}{Q}\right) = \Re\frac{P+Q}{Q},$$

which implies that $P + Q$ does not vanish in Clos $\mathbb{D}$. It follows that

$$B(z) = \tfrac{1}{2}\{P(z) + Q(z)\}$$

does not vanish in Clos $\mathbb{D}$ and satisfies $B(0) = 1$. Next $\deg(P) = \deg(Q) = d+1$ and $P^* = -P$, $Q^* = Q$. Since $P(0) = Q(0) = 1$, this implies that the coefficient at z^{d+1} in B is zero. Passing to the adjoint polynomials, we obtain

$$zB^* = \tfrac{1}{2}\{P(z) - Q(z)\},$$

which proves the corollary. $\square$

Corollary 8.11. *If (P, Q) are alternating polynomials, then*

$$\int_{\mathbb{T}} \log(|P|^2 + |Q|^2)\, dm = \log 4. \tag{71}$$

Proof. It follows from the identity $4|B|^2 = |P|^2 + |Q|^2$, see Corollary 8.10, by Jenssen's formula. □

Theorem 8.12. *Let b be a separable polynomial of degree* $d+1$, $b(0) = 1$, $b^* = b$ *with roots on* $\mathbb{T}$, $0 < \omega < m_b$ *and* $\mathcal{D} = b^2 - 4\omega z^{d+1}$. *Suppose that there is a polynomial r*, $\deg(r) \leqslant d+1$, *such that* $r(0) = 1$, $r^* = -r$ *and*

$$\frac{r}{\sqrt{\mathcal{D}}} \geqslant 1 \quad on\ \{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} < 0\}.$$

Then there is a periodic Schur function f with discriminant $\mathcal{D}$, *which is essentially supported by* $\{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} < 0\}$ *and satisfies*

$$\frac{r}{\sqrt{\mathcal{D}}} = \sigma_{ff^\#}.$$

Proof. Since $4\omega < m_b$, the set $\mathcal{E} = \{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} < 0\}$ is a union of $d+1$ open arcs $\omega_j = (t_j^-, t_j^+)$, $j = 0, 1, \ldots, d$. Let $\gamma_j = [t_j^+, t_{j+1}^-]$. Since $r/\sqrt{\mathcal{D}}$ is positive on ω_j and ω_{j+1}, Lemma 5.5 of [5] (see also [7, Lemma 3.3]) shows that the number of zeros of r in γ_j is odd. The total number of closed arcs γ_j is $d+1$ whereas $\deg(r) \leqslant d+1$. It follows that $\deg(r) = d+1$, r is separable and all roots of r are placed on $\mathbb{T}$ with exactly one root in each γ_j. Hence (r, b) is an alternating pair. By Corollary 8.10 there is a polynomial B, not vanishing in $\operatorname{Clos} \mathbb{D}$ such that $B(0) = 1$ and

$$r = B - zB^*, \quad b = B + zB^*. \tag{72}$$

Since $|r|^2 \geqslant |\mathcal{D}|$ on $\{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} < 0\}$,

$$\bar{z}^{d+1}(\mathcal{D} - r^2) = |r|^2 + \mathcal{T}_{\mathcal{D}} \geqslant 0$$

everywhere on $\mathbb{T}$. It follows that there exists a polynomial A such that

$$\bar{z}^{d+1}(\mathcal{D} - r^2) = |r|^2 + \mathcal{T}_{\mathcal{D}} = |r|^2 + |b|^2 - 4\omega = 4|A|^2 \tag{73}$$

on $\mathbb{T}$. By (72) and (73)

$$4|B|^2 = |r|^2 + |b|^2 = 4|A|^2 + 4\omega$$

on $\mathbb{T}$, which implies that (A, B) is an Wall pair. By (73)

$$\mathcal{D} = (B - zB^*)^2 + 4zA^*A,$$

which shows that the Wall pair (A, B) has the discriminant $\mathcal{D}$. Hence the proof is completed by Theorem 8.5. □

9. Entire quadratic irrationalities

19. It was shown in Section 6 that the family of all periodic Schur functions with a given discriminant $\mathcal{D}$ is controlled by a special algebraic function ρ_1 defined by (55). It does not vanish in $\mathbb{D}$ and (56) followed by the maximum modulus principle implies $1/\rho_1 \in \mathcal{B}$. In fact, ρ_1 is a quadratic irrationality and we can easily write its irreducible quadratic equation. Let x_2 be the second root of (25). Then

$$\rho_2 = \frac{B + zA^*x_2}{\sqrt{\omega}} = \frac{1}{2\sqrt{\omega}}\left\{(B + zB^*) - \sqrt{\mathcal{D}}\right\}. \tag{74}$$

Now, Vieta's Theorem shows that ρ_1 and ρ_2 satisfy the quadratic equation

$$\sqrt{\omega}X^2 - (B + zB^*)X + \sqrt{\omega}z^{d+1} = 0. \tag{75}$$

Since $1/\rho_1 \in \mathcal{B}$ and $\rho_2\rho_1 = z^{d+1}$, we obtain that ρ_2 is an exposed *entire* quadratic irrationality. We also see that $1/\rho_1$ satisfies the quadratic equation

$$\sqrt{\omega}z^{d+1}X^2 - (B + zB^*)X + \sqrt{\omega} = 0. \tag{76}$$

Therefore, $1/\rho_1$ is a quadratic exposed irrationality. However, both ρ_2 and $1/\rho_1$ cannot be periodic (except for case $d = 0$) , since $\sqrt{\omega}$ is a constant.

Theorem 9.1. *The algebraic functions ρ_1 and ρ_2 satisfy*

(a) $$\mathcal{U}\left(\frac{\rho_2}{\rho_1}\right) = \{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} \leqslant 0\}, \tag{77}$$

$$0 < \frac{\rho_2}{\rho_1} < 1 \quad on\ \mathbb{T} \setminus \mathcal{U}\left(\frac{\rho_2}{\rho_1}\right). \tag{78}$$

(b) *ρ_2/ρ_1 is an exposed quadratic irrationality which is a root of the irreducible quadratic equation*

$$\omega z^{d+1}X^2 - \{(B + zB^*)^2 - 2\omega z^{d+1}\}X + \omega z^{1+d} = 0. \tag{79}$$

The discriminant of (79) *is $(B + zB^*)^2\mathcal{D}$.*

(c) *$z^d/\rho_1^2 = \rho_2/z\rho_1$ is the Schur function for the measure $d\varrho$ defined by*

$$\int_{\mathbb{T}} \frac{z+t}{z-t}\, d\varrho(t) = \frac{B + zB^*}{\sqrt{\mathcal{D}}} = \frac{1}{\sqrt{\mathcal{R}}} = \frac{1 + z^{d+1}/\rho_1^2}{1 - z^{d+1}/\rho_1^2}. \tag{80}$$

(d) *For t, $\mathcal{T}_{\mathcal{D}}(t) > 0$*

$$\varrho'(t) = \frac{B + zB^*}{\sqrt{\mathcal{D}}} = \frac{1}{\sqrt{\mathcal{R}}} = \frac{1 + \rho_2/\rho_1}{1 - \rho_2/\rho_1} > 1. \tag{81}$$

(e) *If $\mathcal{D}$ is separable, then $d\varrho$ is absolutely continuous with essential support $\{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}}(t) > 0\}$. If $\mathcal{D}$ has zeros of the second order on $\mathbb{T}$, then ϱ has point masses at these zeros.*

Proof. (a) Using (60) with $\beta \equiv 1$ and (55) with (74), we obtain

$$\frac{\rho_2}{\rho_1} = \frac{(B + zB^*) - \sqrt{\mathcal{D}}}{(B + zB^*) + \sqrt{\mathcal{D}}} = \frac{1 - \sqrt{\mathcal{R}}}{1 + \sqrt{\mathcal{R}}}. \tag{82}$$

On the other hand,

$$\mathcal{R}(z) = \frac{\mathcal{D}(z)}{(B + zB^*)^2} = \frac{\mathcal{T}_{\mathcal{D}}(z)}{|B + zB^*|^2}, \tag{83}$$

which shows that for $z \in \mathbb{T}$

$$\sqrt{\mathcal{R}(z)} = \frac{\sqrt{\mathcal{T}_{\mathcal{D}}(z)}}{|B + zB^*|}. \tag{84}$$

Observing that $\sqrt{\mathcal{R}}$ is purely imaginary on $\{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} \leqslant 0\}$ and real on the complement to this set, we obtain (77) from (82). Since

$$\frac{\rho_2}{\rho_1} = \frac{\rho_2\rho_1}{\rho_1^2} = \frac{z^{d+1}}{\rho_1^2} \in \mathcal{B}, \tag{85}$$

the quotient $|\rho_2/\rho_1|$ cannot exceed 1 in Clos $\mathbb{D}$. Therefore, (82) shows that $\sqrt{\mathcal{R}}$ must be positive on the subset of $\mathbb{T}$, where $|\rho_2/\rho_1| < 1$. This implies again by (82) that $0 < \rho_2/\rho_1 < 1$ as soon as $|\rho_2/\rho_1| < 1$ on $\mathbb{T}$. Notice that by (74) ρ_2 cannot vanish on $\mathbb{T}$.

(b) To find the quadratic equation for ρ_2/ρ_1 we compute

$$\operatorname{Tr}\left(\rho_2/\rho_1\right) = \frac{\rho_2}{\rho_1} + \frac{\rho_1}{\rho_2} = \frac{(\rho_1+\rho_2)^2}{\rho_1\rho_2} - 2 = \frac{(B+zB^*)^2}{\omega z^{g+1}} - 2.$$

Observing that the norm of ρ_2/ρ_1 is 1, we obtain (79).

(c) It follows from (81).

(d) It follows from (78) and Fatou's theorem.

(e) It follows from (80). Just observe that by (85) the Schur function of ϱ multiplied by z equals 1 exactly at the points $\rho_1 = \rho_2$, i.e. at the zeros of $\mathcal{D}$. $\square$

Elementary calculations show that

$$\frac{\rho_2}{z^{d+1}\rho_1} = \frac{1}{\rho_1^2} = \frac{\omega}{(B+zA^*f)^2}. \tag{86}$$

Notice that the identity on $\mathbb{T}$

$$1 - |f(z)|^2 = \frac{\omega(1-|f(z)|^2)}{|B+zA^*f|^2}$$

is in a complete agreement with Theorem 9.1 (a).

Corollary 9.2. *Let (A, B) be an Wall pair such that its discriminant $\mathcal{D}$ is separable. Then the system $\{1, \zeta, \ldots, \zeta^d\}$ is orthogonal with respect to the weight*

$$\frac{B+zB^*}{\sqrt{\mathcal{D}}}$$

on the set $\{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} > 0\}$.

Proof. Apply Theorem 9.1 (c) and (d). $\square$

20. Wall Pairs in Corollary 9.2 provide explicit formulas for curious functions in Harmonic Analysis on the unit circle. By (81) the function

$$u = \Re\left(\frac{B+zB^*}{\sqrt{\mathcal{D}}}\right)$$

is positive on $G = \{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} > 0\}$ and vanishes on $\mathbb{T} \setminus G$ except for a finite number of end-points of this set. It is clear that $u \in L^p(\mathbb{T})$ for every $p < 2$, whereas the harmonically conjugate function $v = \widetilde{u}$ is supported by the set $\mathbb{T} \setminus G$.

21. By Theorem 3.3 the roots of the discriminant $\mathcal{D}$ of a periodic exposed quadratic irrationality ρ_2 are either simple or of the second order and are placed on $\mathbb{T}$. As it was shown the case of multiple roots of order 2 is the limit case of separable discriminants. Therefore, it is advantageous technically to consider separable discriminants passing to the limit if necessary. We saw that Wall pairs supported by a given set $\mathcal{E} = \{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} < 0\}$ are controlled by exposed entire irrationality ρ_2. By [5, Lemma 5.2] an exposed quadratic irrationality in $\mathcal{B}$ satisfies the quadratic equation

$$\sqrt{\omega}X^2 + bX + \overline{\lambda}\sqrt{\omega}z^{1+d} = 0, \tag{87}$$

where $\deg(b) \leqslant d + 1$, $b^* = \lambda b$, $\omega > 0$. If $\deg(b) < d + 1$, then easy calculation shows that $b = z^s b_1$ with $b_1(0) \neq 0$, $\deg(b_1) = d + 1 - 2s$:

$$z^{1+d}\overline{b} = z^{1+d-s}\overline{b}_1 = \lambda z^s b_1.$$

If $f \in \mathcal{B}$ satisfies (87), then counting orders of zero at $z = 0$ shows that $f = z^s g$, $g(0) \neq 0$, where g is an exposed entire irrationality satisfying (87) with $b := b_1$, $d + 1 := d + 1 - 2s$. It follows that one can consider the case $|b(0)| = 1$ in (87). The explanation of this fact follows from the observation that $z^s f$ is an exposed entire quadratic rationality as soon as s is a positive integer and f is an exposed quadratic irrationality.

The following theorem shows that if one assumes that the discriminant of (87) is a separable polynomial of degree $2d + 2$, then all roots of b lie on $\mathbb{T}$.

Theorem 9.3. *Let $b^* = \lambda b$ be a polynomial of degree $d + 1$ such that the equation $|b| = c$ has $2d + 2$ different zeros on $\mathbb{T}$ for some $c > 0$. Then b is a separable polynomial with roots on $\mathbb{T}$.*

Proof. Applying rotations, we may assume without loss of generality that $\lambda = 1$, see [5, formula (36)].

Case 1: Suppose that $d + 1 = 2k$ with integer k. Then $b^* = b$ implies that the trigonometric polynomial $t(z) = \bar{z}^k b(z)$ is real. Since $|b(z)| = |t(z)|$ on $\mathbb{T}$, the equation $|t(z)| = c$ has $2d + 2$ roots $z_1, z_2, \ldots, z_{2d+2}$ on $\mathbb{T}$ numbered counterclockwise. At every point z_j we have $t(z_j) = \pm c$. Hence every open arc (z_j, z_{j+1}) contains either a root of $\dot{t}$ or a root of t. The total number of open arcs (z_j, z_{j+1}) is $2d + 2$. The number of roots of every polynomial t and $\dot{t}$ cannot exceed $d + 1$. Hence all roots of t, and consequently of b, are placed on $\mathbb{T}$.

Case 2: Suppose that $d = 2k$ with integer k. Then

$$z^{k+1/2}\overline{b}(z) = \bar{z}^{k+1/2} b(z) = t(z), \quad z \in \mathbb{T}$$

is a real function. Here the root $\sqrt{z}$ is chosen by a cut passing through a point ζ not equal to any root $\{z_1, \ldots, z_{2d+2}\}$ of $|t(z)| = c$. Every arc (z_j, z_{j+1}) not containing ζ either contains a zero of t or a zero of $\dot{t}$. The zeros of t coincide with zeros of b. Zeros of $\dot{t}$ coincide with zeros of

$$\dot{t} = e^{-i(\theta/2+k\theta)}(-i/2 - ik)\, b + e^{-i(\theta/2+k\theta)}\,\dot{b}.$$

Since these zeros are the zeros of the polynomial

$$\dot{b} - i\left(k + \frac{1}{2}\right) b,$$

the number of these zeros cannot exceed its degree $d + 1$. Consequently, $2d + 1$ arc contain $2d + 1$ zeros of t and $\dot{t}$. Since $\dot{t}$ may have at most $d + 1$ zeros, t must have at least $2d + 1 - (d + 1) = d$

zeros on $\mathbb{T}$. But b is self-adjoint. This means that the zeros which are not on $\mathbb{T}$, must enter it in pairs symmetric with respect to $\mathbb{T}$. But in our case only one zero remains. Hence it must be on $\mathbb{T}$. □

10. Periodic measures with a given essential support

22. Any Wall pair (A, B) determines two polynomials b_+ and b_- of degree $d+1$ with roots on $\mathbb{T}$

$$b_+ = B + zB^*, \quad b_- = B - zB^*. \tag{88}$$

These polynomials satisfy

$$\begin{aligned} b_+^* &= b, \quad b_+(0) = 1, \\ b_-^* &= -b, \quad b_-(0) = 1. \end{aligned} \tag{89}$$

On $\mathbb{T}$ polynomials b_+ and b_- are related by the formula

$$4|B|^2 = |b_+|^2 + |b_-|^2 = 4|A|^2 + 4\omega. \tag{90}$$

Polynomial b_+ and the parameter ω determine the essential support $\mathcal{E}(f)$ of σ. From the purpose of understanding which periodic measures can be carried by $\mathcal{E}(f)$, it is important to describe all solutions in B of the equation

$$B + zB^* = b_+. \tag{91}$$

Lemma 10.1. *All polynomials B satisfying* (91) *and not vanishing in the closed disc are obtained by the formula*

$$\frac{B - zB^*}{B + zB^*} = \int_{b_+(\zeta)=0} \frac{\zeta + z}{\zeta - z}\, d\tau, \tag{92}$$

where τ varies the set of probability measures distributed through the zero set of b_+ so that each root of b_+ carries a positive mass.

Proof. Any measure τ distributed as it is said in the lemma has a Blaschke product $f = \lambda\varphi^*/\varphi$ as its Schur function

$$\int_{b_+(\zeta)=0} \frac{\zeta + z}{\zeta - z}\, d\tau = \frac{1 + zf}{1 - zf} = \frac{\varphi + \lambda z\varphi^*}{\varphi - \lambda z\varphi^*}.$$

The requirement that $\tau(s) > 0$ for every zero of b_+ implies that the polynomial $\varphi - \lambda z\varphi^*$ has the same zero set as b_+. Observing that $\varphi(0) = b_+(0) = 1$, we obtain that $b_+ = \varphi - \lambda z\varphi^*$. Now, taking into account that $b_+^* = b_+$, we conclude that

$$b_+ = (\varphi - \lambda z\varphi^*)^* = z\varphi^* - \bar{\lambda}B = -\bar{\lambda}(\varphi - \lambda z\varphi^*) = -\bar{\lambda}b. \tag{93}$$

It follows that $-\lambda = 1$. □

Thus, any choice of a probability measure τ gives a polynomial B of degree d, $B(0) = 1$ satisfying

$$\begin{aligned} B - zB^* &= b_+ \int_{b_+(\zeta)=0} \tfrac{\zeta+z}{\zeta-z}\, d\tau \\ B + zB^* &= b_+, \end{aligned} \tag{94}$$

and we obtain explicit formulas for B and zB^*

$$\begin{aligned} B \quad &= \tfrac{b_+}{2} \left\{ 1 + \int_{b_+(\zeta)=0} \tfrac{\zeta+z}{\zeta-z}\, d\tau \right\}, \\ zB^* &= \tfrac{b_+}{2} \left\{ 1 - \int_{b_+(\zeta)=0} \tfrac{\zeta+z}{\zeta-z}\, d\tau \right\}. \end{aligned} \tag{95}$$

Corollary 10.2. *Given any polynomial b with all roots on $\mathbb{T}$ such that $b^* = b$ and $b(0) = 1$ and a sufficiently small positive number $\omega < 1$ there exists an Wall pair (A, B) with the parameter ω such that $B + zB^* = b$. In particular, the set of arcs $\mathcal{E}(b, \omega) = \{t \in \mathbb{T} : |b|^2 - 4\omega < 0\}$ is an essential support for periodic measures for sufficiently small ω.*

Proof. It follows from (95) since

$$|B|^2 = \frac{1}{4}|b_+|^2 \left\{ 1 + \left| \int_{b_+(\zeta)=0} \frac{2\Im(\zeta\bar{z})}{|\zeta - z|^2}\, d\tau \right|^2 \right\} > 0 \tag{96}$$

on $\mathbb{T}$. $\square$

This demonstrates an interesting phenomenon. Smaller the set $\mathcal{E}(b, \omega)$ is, i.e. smaller ω is, bigger is the family of periodic Schur functions essentially supported by $\mathcal{E}(b, \omega)$.

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